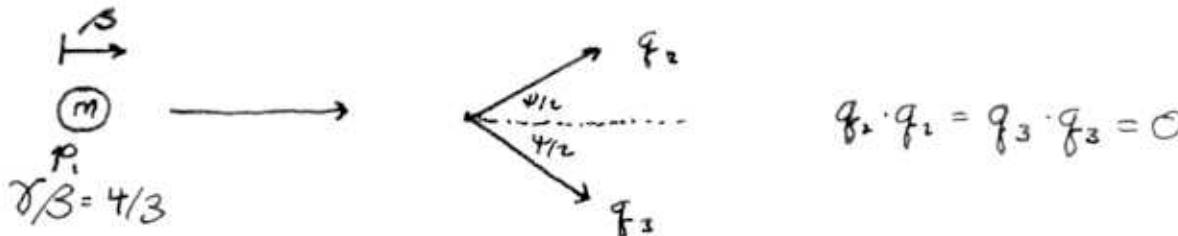


Physics 110B
Homework #5

#1. (a)



$$t = \gamma \tau$$

$$x = \beta c t = \beta c \gamma \tau = \boxed{4/3 c \tau = x}$$

$$(b) \quad p_i = (\gamma m c, \gamma \beta m c, 0, 0) \quad q_1 = (q_0, q_0 \cos \psi_{1/2}, q_0 \sin \psi_{1/2}, 0)$$

$$q_2 = (q_0, q_0 \cos \psi_{1/2}, -q_0 \sin \psi_{1/2}, 0)$$

Energy Conservation:

$$\gamma m c^2 = (q_0 + q_0)c \Rightarrow \underline{q_0 = \frac{1}{2} \gamma m c}$$

Momentum Conservation:

$$\gamma \beta m c = 2 q_0 \cos \psi_{1/2} = \gamma m c \cos \psi_{1/2} \Rightarrow \underline{\cos \psi_{1/2} = \beta}$$

Energy-Momentum Conservation:

$$p_i = q_2 + q_3$$

$$p_i \cdot p_i = q_2 \cancel{q_2} + 2 q_2 \cdot q_3 + q_3 \cancel{q_3}$$

$$m^2 c^2 = 2 q_0^2 (1 - \cos^2 \psi_{1/2} + \sin^2 \psi_{1/2})$$

$$m^2 c^2 = \frac{1}{2} \gamma^2 m^2 c^2 (2 \sin^2 \psi_{1/2})$$

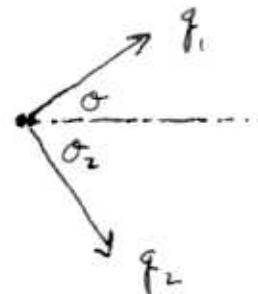
$$\underline{\frac{1}{\gamma} = \sin \psi_{1/2}}$$

$$\tan \psi_2 = \frac{\sin \psi_2}{\cos \psi_2} = \frac{1}{\gamma \beta} = \frac{3}{4}$$

$$\psi_2 = \arctan \frac{3}{4} = 36.87^\circ$$

$$\Rightarrow \boxed{\psi = 73.7^\circ}$$

#2. $\begin{array}{c} \text{①} \\ \text{E} = \gamma m \\ p_1 \end{array} + \begin{array}{c} \text{②} \\ p_2 \end{array} \longrightarrow$



$$q_1 \cdot q_1 = q_2 \cdot q_2 = 0$$

$$p_1 = (\gamma m c, \gamma \beta m c, 0, 0)$$

$$q_1 = (E'_1, E'_1 \cos \theta, E'_1 \sin \theta, 0)$$

$$p_2 = (m c, 0, 0, 0)$$

$$q_2 = (E'_2, E'_2 \cos \theta_2, E'_2 \sin \theta_2, 0)$$

$$p_1 + p_2 = q_1 + q_2$$

$$p_1 + p_2 - q_1 = q_2$$

$$p_1 \cdot p_1 + p_2 \cdot p_2 + q_1 \cdot q_1 + 2p_1 \cdot p_2 - 2p_1 \cdot q_1 - 2p_2 \cdot q_1 = q_2 \cdot q_2$$

$$m^2 c^2 + m^2 c^2 + 0 + 2\gamma m^2 c^2 = 2(\gamma \frac{E'}{c} m c - \gamma \beta m c \frac{E'}{c} \cos \theta) + 2 \frac{E'}{c} m c$$

$$\Rightarrow 2(1 + \gamma) m^2 c^2 = 2 \frac{E'}{c} m c (1 + \gamma - \gamma \beta \cos \theta)$$

$$\frac{m c^2}{E'} = 1 - \frac{\gamma \beta}{1 + \gamma} \cos \theta$$

$$\frac{m c^2}{E'} = 1 - \frac{\gamma (1 - \gamma \beta^2)^{1/2}}{1 + \gamma} \cos \theta$$

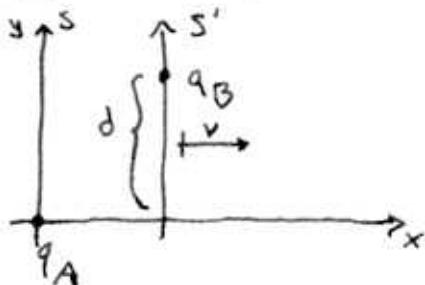
$$= 1 - \frac{\gamma \sqrt{\gamma} (\gamma^2 - 1)^{1/2}}{1 + \gamma} \cos \theta$$

$$= 1 - \frac{((\gamma-1)(\gamma+1))^{1/2}}{\gamma+1} \cos\alpha$$

$$\Rightarrow \boxed{\frac{mc^2}{E'} = 1 - \sqrt{\frac{\gamma-1}{\gamma+1}} \cos\alpha}$$

#3. (Griffiths 12.44)

(a)



$$\text{Fields of } A \text{ at } B: \bar{E} = \frac{1}{4\pi\epsilon_0} \frac{q_A}{d^2} \hat{y}; \quad \bar{B} = 0$$

So, the force on q_B is

$$\bar{F} = \frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{d^2} \hat{y}$$

$$(b) (i) F_y = \frac{dp_y}{dt} = \frac{dp'_y}{\gamma dt' + \beta B/c dx'} = \frac{\partial p'_y / \partial t'}{\gamma(1 - \beta \frac{dx'}{\partial t'})}$$

$$= \frac{F'_y}{\gamma(1 - \beta \frac{dx'}{c})} = F'_y / \gamma$$

$$\Rightarrow F'_y = \gamma F_y = \boxed{\frac{\gamma}{4\pi\epsilon_0} \frac{q_A q_B}{d^2} \hat{y} = F'}$$

$$(ii) E'_y = \gamma(E_y - \nu B_z) \quad (\text{eq 12.108})$$

$$E'_y = \gamma \frac{1}{4\pi\epsilon_0} \frac{q_A}{d^2} \hat{y}$$

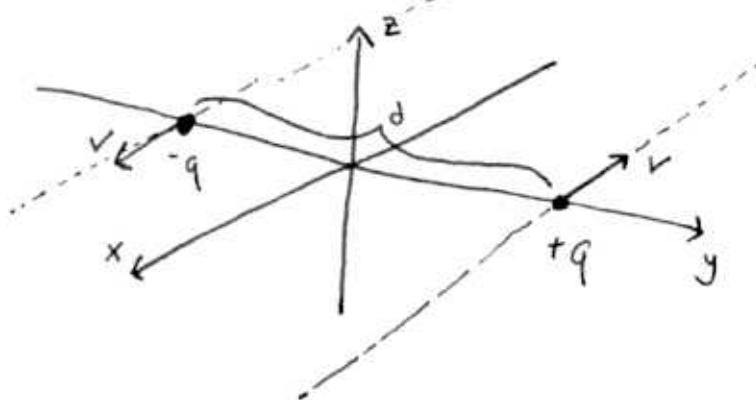
$$B'_y = \gamma(B_y + \nu c^2 E_z) = 0$$

$$E_z' = \gamma(E_z + vB_y) = 0$$

$$B_z' = \gamma(B_z - \frac{v}{c^2}E_y) = \gamma \frac{v}{c^2} \frac{1}{4\pi\epsilon_0} \frac{q_A}{r^2} \hat{y}$$

$$\bar{F}' = q_B (\bar{E}' + \vec{v} \times \vec{B}') = \boxed{q_B \frac{\gamma}{4\pi\epsilon_0} \frac{q_A}{r^2} \hat{y} = F'}$$

#4. (Griffiths 12.45),



We will start with System C (-q at rest) and then transform into the various other systems. The check will be $E^2 - c^2 B^2$ since this is a Lorentz invariant.

System C (-q at rest):

$$V_c = 0 \quad \underline{\bar{E} = -\frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{y}} ; \quad \underline{\bar{B} = 0} ; \quad \underline{\bar{F} = \bar{q} \bar{E} = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{r^2} \hat{y}}$$

$$\text{Check: } E^2 - c^2 B^2 = \frac{q^2}{(4\pi\epsilon_0)^2} \frac{1}{r^4}$$

System B (+q at rest):

$$V_B = \frac{V + V}{1 + \frac{V^2}{c^2}} = \frac{2V}{1 + V^2/c^2}$$

$$\gamma_B = \left(1 - \frac{4V^2/c^2}{(1 + V^2/c^2)^2} \right)^{1/2} = \frac{1 + V^2/c^2}{\left(1 - \frac{2V^2}{c^2} + \frac{V^4}{c^4} \right)^{1/2}}$$

$$= \frac{1 + v^2/c^2}{((1 - v^2/c^2)^2)^{1/2}} = \frac{1 + v^2/c^2}{1 - v^2/c^2} = \gamma^2(1 + \frac{v^2}{c^2})$$

$$\gamma = (1 - v^2/c^2)^{-1/2}$$

Thus, by Lorentz transforming :

$$\bar{E} = -\gamma_B \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{y} = -\frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \gamma^2(1 + \frac{v^2}{c^2}) \hat{y}$$

$$\begin{aligned} \bar{B} &= \gamma_B \frac{v}{c^2} E_y = -\gamma^2(1 + \frac{v^2}{c^2}) \frac{2v/c^2}{1 + v^2/c^2} \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{z} \\ &= -\frac{\gamma^2 2v/c^2}{4\pi\epsilon_0} \frac{q}{r^2} \hat{z} \end{aligned}$$

$$\text{Check: } E^2 - c^2 B^2 = \left(\frac{q}{4\pi\epsilon_0} \frac{\hat{y}}{r^2}\right)^2 \gamma^4 (1 + \frac{v^2}{c^2})^2 - c^2 \gamma^4 \frac{4v^2}{c^4} \left(\frac{q}{4\pi\epsilon_0} \frac{\hat{z}}{r^2}\right)^2$$

$$= \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2 \left(\gamma^4 \left((1 + \frac{v^2}{c^2})^2 - 4v^2/c^2 \right) \right)$$

$$= \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2 \left(\gamma^4 \left(1 - 2v^2/c^2 + v^4/c^4 \right) \right)$$

$$= \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2 \left(\gamma^4 (1 - \frac{v^2}{c^2})^2 \right) = \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2$$

$$\bar{F} = q(\bar{E} + v \times \bar{B}) \quad v=0 \text{ for } +q \text{ at rest}$$

$$= q \bar{E} = \frac{q^2}{4\pi\epsilon_0} \frac{1}{r^2} \gamma^2 (1 + \frac{v^2}{c^2}) \hat{y}$$

System A (Figure given at the beginning of the problem)

$$V_A = v \hat{x}$$

$$\gamma_A = \gamma$$

$$\bar{E} = -\gamma_A \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{y} = -\frac{q}{4\pi\epsilon_0} \frac{\gamma}{r^2} \hat{y}$$

$$\bar{B} = \gamma_A v_A/c^2 E_y = -\frac{v}{c^2} \frac{q}{4\pi\epsilon_0} \frac{\gamma}{r^2} \hat{z}$$

$$\bar{F} = q(\bar{E} + \nabla \times \bar{B}) = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{r^2} (\gamma \hat{y} - v \frac{v^2}{c^2} (\hat{x} \times \hat{z}))$$

$$= -\frac{q^2}{4\pi\epsilon_0} \frac{\gamma}{r^2} \left(1 + \frac{v^2}{c^2}\right) \hat{y}$$

Check: $E^2 - c^2 B^2 = \left(-\frac{q}{4\pi\epsilon_0} \frac{\hat{y}}{r^2}\right)^2 \left(\gamma^2 - c^2 \frac{v^2}{c^4} \gamma^2\right)$

$$= \left(-\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2 \gamma^2 \left(1 - \frac{v^2}{c^2}\right) = \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2$$

Placing the results together in a table we have:

	System A	System B	System C
E at $+q$ due to $-q$	$-\frac{q}{4\pi\epsilon_0 r^2} \gamma \hat{y}$	$-\frac{q}{4\pi\epsilon_0 r^2} \gamma^2 \left(1 + \frac{v^2}{c^2}\right) \hat{y}$	$-\frac{q}{4\pi\epsilon_0 r^2} \hat{y}$
B at $-q$ due to $-q$	$-\frac{q}{4\pi\epsilon_0 r^2} \frac{v}{c^2} \gamma \hat{z}$	$-\frac{q}{4\pi\epsilon_0 r^2} \frac{2v}{c^2} \gamma^2 \hat{z}$	0
F on $+q$ due to $-q$	$-\frac{q^2}{4\pi\epsilon_0 r^2} \gamma \left(1 + \frac{v^2}{c^2}\right) \hat{y}$	$-\frac{q^2}{4\pi\epsilon_0 r^2} \gamma^2 \left(1 + \frac{v^2}{c^2}\right) \hat{y}$	$-\frac{q}{4\pi\epsilon_0 r^2} \hat{y}$

#5. (Griffiths 12.46)

(a) Using (Eq. 12.108):

$$\bar{E}' \cdot \bar{B}' = E'_x B'_x + E'_y B'_y + E'_z B'_z$$

$$= E_x B_x + \gamma^2 (E_y - v B_z)(B_y + v/c^2 E_z) + \gamma (E_z + v B_y)(B_z - v/c^2 E_y)$$

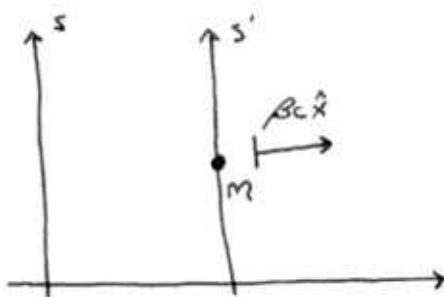
$$= E_x B_x + \gamma^2 (E_y B_y + v/c^2 E_y E_z - v B_x B_z + E_z B_z - v^2/c^2 E_z B_z)$$

$$\begin{aligned}
 & -\sqrt{c^2} E_y B_z + \sqrt{B_y} B_z - \sqrt{\frac{c^2}{c^2}} E_y B_y \\
 & = E_x B_x + \gamma^2 (E_y B_y (1 - \sqrt{\frac{c^2}{c^2}}) + E_z B_z (1 - \sqrt{\frac{c^2}{c^2}})) \\
 & = E_x B_x + E_y B_y + E_z B_z \\
 \boxed{E' \cdot \bar{B}' = \bar{E} \cdot \bar{B}} \quad & \text{QED}
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad E'^2 - c^2 B'^2 &= (E_x^2 + \gamma^2(E_y - \sqrt{B_z})^2 + \gamma^2(E_z + \sqrt{B_y})^2) \\
 &\quad - c^2(B_x^2 + \gamma^2(B_y + \frac{\sqrt{c^2}}{c^2} E_z)^2 + \gamma^2(B_z - \frac{\sqrt{c^2}}{c^2} E_y)^2) \\
 &= E_x^2 + \gamma^2(E_y^2 - 2E_y \sqrt{B_z} + \sqrt{B_z}^2 + E_z^2 + 2E_z \sqrt{B_y} + \sqrt{B_y}^2) \\
 &\quad - c^2 B_y^2 - c^2 2 \cancel{\sqrt{c^2}} B_y E_z - c^2 \cancel{\frac{c^2}{c^4}} E_z^2 - c^2 B_z^2 + c^2 2 \cancel{\sqrt{c^2}} B_z E_y \\
 &\quad - c^2 \cancel{\frac{c^2}{c^4}} E_y^2) - c^2 B_x^2 \\
 &= E_x^2 - c^2 B_x^2 + \gamma^2(E_y^2 (1 - \sqrt{\frac{c^2}{c^2}}) + E_z^2 (1 - \sqrt{\frac{c^2}{c^2}}) - c^2 B_y^2 (1 - \sqrt{\frac{c^2}{c^2}}) \\
 &\quad - c^2 B_z^2 (1 - \sqrt{\frac{c^2}{c^2}})) \\
 &= (E_x^2 + E_y^2 + E_z^2) - c^2 (B_x^2 + B_y^2 + B_z^2) \\
 \boxed{E'^2 - c^2 B'^2 = E^2 - c^2 B^2} \quad & \text{QED}
 \end{aligned}$$

(c) No, it is not possible to find another system in which the electric field is zero at P. For if $B=0$ in one system, then $(E^2 - c^2 B^2)$ is positive. Since it is invariant, it must be positive in any system. Therefore $E \neq 0$ in all systems.

#6.



$$\rho_\mu h^\mu = 0$$

$$\rho_\mu = (\gamma m c, \gamma \beta m c, 0, 0) \quad h^\mu = (h_0, h_1, h_2, h_3)$$

$$\rho'_\mu = (m c, 0, 0, 0) \quad h'^\mu = (h'_0, h'_1, h'_2, h'_3)$$

$$\rho'_\mu h'^\mu = m c h'_0 = 0 \Rightarrow \underline{h'_0 = 0}$$

$$\rho_\mu h^\mu = \gamma m c h_0 - \gamma \beta m c h_1 = 0 \Rightarrow \underline{h_0 = \beta h_1}$$

Doing a Lorenz transformation:

$$h'_0 = \gamma (h_0 - \beta h_1) = 0$$

$$h'_1 = \gamma (-\beta h_0 + h_1) = \gamma (-\beta^2 + 1) h_1 = h_1 / \gamma$$

$$h'_2 = h_2$$

$$h'_3 = h'_3$$

Thus,

$$\boxed{h_0 = \gamma h'_1; \quad h_0 = \gamma \beta h'_1}$$

$$h_2 = h'_2; \quad h_3 = h'_3$$

$$\#7. \quad h_\mu \equiv g_{\mu\nu} h^\nu; \quad h_\mu h^\mu = \text{invariant length squared}$$

(a)

$$\boxed{g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}$$

$$(b) T_{\mu\nu} = g_{\mu\alpha} T^{\alpha\beta} g_{\beta\nu}$$

Using the above transformation:

$$g_{\mu\nu} = g_{\mu\alpha} g^{\alpha\beta} g_{\beta\nu}$$

$$\textcircled{1} \quad g_{\mu\alpha}^{-1} g_{\mu\nu} g_{\nu\alpha}^{-1} = g_{\mu\alpha}^{-1} g_{\mu\alpha} g^{\alpha\beta} g_{\beta\nu} g_{\nu\alpha}^{-1}$$

$$g_{\mu\alpha}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} = g_{\mu\alpha}$$

Thus eq \textcircled{1} becomes:

$$\begin{aligned} g^{\alpha\beta} &= g_{\mu\alpha}^{-1} g_{\mu\nu} g_{\nu\beta}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} = g^{\alpha\beta} \end{aligned}$$

Thus,

$$\boxed{g^{\mu\nu} = g_{\mu\nu}}$$

(c) We begin with the result from part (a)

$$g_{\mu\alpha} = g^{\mu\alpha} \Rightarrow g_{\mu\alpha} (g^{\nu\alpha})^{-1} = g^{\mu\alpha} (g^{\nu\alpha})^{-1}$$

$$\Rightarrow g_{\mu\alpha} g^{\alpha\nu} = \underbrace{g^{\mu\alpha} (g^{\nu\alpha})^{-1}}_{\delta_\mu^\nu}$$

$$\text{Thus, } \boxed{g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu^\nu}$$

#8. $H^{\mu\nu}$ is antisymmetric w/ 6 independent components.

$$H^{\mu\nu} = \begin{pmatrix} 0 & H^{01} & H^{02} & H^{03} \\ H^{10} & 0 & H^{12} & H^{13} \\ H^{20} & H^{21} & 0 & H^{23} \\ H^{30} & H^{31} & H^{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & H^{01} & H^{02} & H^{03} \\ -H^{01} & 0 & H^{12} & H^{13} \\ -H^{02} & -H^{12} & 0 & H^{23} \\ -H^{03} & -H^{13} & -H^{23} & 0 \end{pmatrix}$$

$$\text{Using: } H_{\mu\nu} = g_{\alpha\mu} H^{\alpha\beta} g_{\beta\nu}$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} H^{\alpha\beta} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -H^{01} & -H^{02} & -H^{03} \\ -H^{01} & 0 & -H^{12} & -H^{13} \\ -H^{02} & H^{12} & 0 & H^{23} \\ -H^{03} & H^{13} & H^{23} & 0 \end{pmatrix}$$

$$\boxed{H_{\mu\nu} = \begin{pmatrix} 0 & -H^{01} & -H^{02} & -H^{03} \\ H^{01} & 0 & H^{12} & H^{13} \\ H^{02} & -H^{12} & 0 & H^{23} \\ H^{03} & -H^{13} & -H^{23} & 0 \end{pmatrix}}$$